Journal of Statistical Physics, Vol. 36, Nos. 5/6, 1984

Is the Dimension of Chaotic Attractors Invariant under Coordinate Changes?

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Several different dimensionlike quantities, which have been suggested as being relevant to the study of chaotic attractors, are examined. In particular, we discuss whether these quantities are invariant under changes of variables that are differentiable except at a finite number of points. It is found that some are and some are not. It is suggested that the word "dimension" be reversed only for those quantities have this invariance property.

KEY WORDS: Chaotic attractors; coordinate changes; invariance.

1. INTRODUCTION

Recently, researchers in many fields of science have shown that objects with fractional dimension[1] play an important, sometimes crucial, role in the problems they consider. In these various research contributions, one finds a variety of definitions of dimension. In particular, there exist concepts of dimension for a set in a metric space and, in addition, concepts of dimension for a probability measure in a metric space. (A probability measure is one for which the measure of the entire space is 1.) See Farmer *et al.*[2] for a discussion and review of metric and measure dimensions within the context of chaotic attractors. As an example of a metric space dimension we mention the Hausdorff dimension (a definition of Hausdorff dimension is given in Section 4). As an example of a probability measure dimension we give, below, the definition of the information dimension,[3] which we denote d_I.

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0022-4715/84/0900-0687$03.50/0

Definition 1.

$$d_I(\mu) = \lim_{\varepsilon \to 0} \frac{\sum_{i=1}^{N(\varepsilon)} p_i \ln p_i}{\ln \varepsilon} \tag{1.1}$$

where the support of the measure μ has been covered by $N(\varepsilon)$ cubes of edge length ε, and p_i denotes the total probability measure within the ith such cube.

For the various definitions of dimensions of a probability measure considered in Ref. 2, Farmer *et al.* found that all these take on a common value for the examples of chatoic attractors given in that paper. Thus they conjecture that this is true in general, and, based upon this equality of dimensions for probability measures, they call their common value "the dimension of the measure." More recently, Hentschel and Procacia[4] and Grassberger[5] have introduced certain scaling exponents for a measure in a metric space. These scaling exponents, which we denote d_q, depend on a continuous parameter q, and, in general, are unequal for different values of q. Thus, if d_q is admitted to be a probability measure dimension, then probability measure dimensions can take on infinitely many different values for a single measure and there is no single "dimension of the measure." This situation brings into focus the question of what we mean by a dimension. While we do not here propose a general consistent answer to this question, we do propose a requirement that a quantity should satisfy in order that it be called a dimension; in particular, we require that all dimensions be invariant under "reasonable" changes of variables. It will be shown in this paper that the d_q fail this test for $q \neq 1$, and so, according to our criterion, should not be called dimensions. On the other hand, for the coordinate changes we consider, both the information dimension and the Hausdorff dimension are invariant.

The scaling exponents d_q introduced in Refs. 4 and 5 are given by

$$d_q(\mu) = \frac{1}{q-1} \lim_{\varepsilon \to 0} \frac{\ln \sum_{i=1}^{N(\varepsilon)} p_i^q}{\ln \varepsilon} \tag{1.2}$$

Thus for $q = 0$ we recover the definition of the "capacity" of a set,

$$d_0(\mu) = d_0(V) = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \tag{1.3}$$

where $N(\varepsilon)$ is simply the number of ε cubes needed to cover the support set V. [Note that (1.3) is defined for any set (when the limit exists) and so does not require a probability measure (i.e., it does not depend on the p_i);

hence, d_0 might be considered as a candidate for being a metric dimension. However, since it fails the coordinate change test, the capacity is not a proper metric dimension.] In addition, Hentschel and Procaccia[4] show that the information dimension can be obtained from d_q by taking the limit as q aprpoaches one in Eq. (1.2),

$$d_I = \lim_{q \to 1} d_q$$

Furthermore, they show that the d_q for $q \neq 1$ can be used to obtain useful bounds on d_I.

This paper is organized as follows. In Section 2 we consider the case $q > 1$ and present examples of maps with attractors which have a smooth invariant probability density and for which d_q can be altered by fairly simple changes of variables. In Section 3 we consider the case of $q < 1$ and show that d_q for a Cantor set is not invariant under coordinate changes. Finally, in Section 4 it is shown that the Hausdorff dimension and information dimension are invariant for the type of variable changes considered in Sections 2 and 3.

Based on these results we believe that there are only two proper types of dimension definitions appropriate for attractors, metric dimensions (Hausdorff dimension) and probability measure dimensions, and that members of these classes of dimension definitions take on a value which is the same for all members of the class (of course, the values of the metric dimension may be different from the value of the probability measure dimension).

To conclude this section, we comment on the type of "reasonable" variable changes used in Sections 2–4. These variable changes are invertible, and they have a derivative which is nonzero and finite except at a finite number of points. In particular, the variables changes considered do have infinite derivatives at certain points. (The d_q are invariant for changes of variables when the derivative and its inverse are both uniformly bounded.)

2. CHANGE OF d_q UNDER VARIABLE TRANSFORMATION FOR $q > 1$

As a simple example consider the logistic map,

$$x_{n+1} = 4x_n(1 - x_n)$$

For almost any initial condition $x_0 \in [0, 1]$, this map produces an invariant density

$$f_1(x) = \frac{1}{\pi[x(1 - x)]^{1/2}}$$

for $x \in [0, 1]$. Now introduce the change of coordinates[6]

$$y = \int_0^x f_1(x')\,dx' = \frac{1}{2} - \frac{\sin^{-1}(1-2x)}{\pi}$$

with $\sin^{-1}$ defined to be $\pi/2 \geqslant \sin^{-1}(1-2x) \geqslant -\pi/2$ for $x \in [0, 1]$. Using this change of variables, the original logistic map is transformed to the well-known "tent map,"

$$y_{n+1} = \begin{cases} 2y_n, & \text{for} \quad y_n \leqslant 1/2 \\ 1 - 2y_n, & \text{for} \quad y_n \geqslant 1/2 \end{cases}$$

which for almost every initial condition $y_0 \in [0, 1]$ produces the invariant density,

$$\bar{f}(y) = 1$$

for $y \in [0, 1]$. Thus $p_i = \varepsilon$ in Eq. (1.2) and $d_q = 1$ for all q for the density $\bar{f}(y)$.

Now consider d_q corresponding to $f_1(x)$. Divide the interval $[0, 1]$ into $2K$ cells of length $\varepsilon = 1/(2K)$, where K is an integer. The fraction of the measure in the ith cell is

$$p_i = \int_{(i-1)\varepsilon}^{i\varepsilon} f_1(x)\,dx$$

For $1/2 \geqslant i\varepsilon > 0$,

$$p_i \cong \frac{2}{\pi}\,\varepsilon^{1/2}[i^{1/2} - (i-1)^{1/2}]$$

and we have the estimate

$$p_i \sim (\varepsilon/i)^{1/2} \qquad \text{for} \quad i = 1, \dots, K$$

Thus, since $f_1(x)$ is symmetric about $x = 1/2$, for $q > 1$, $q \neq 2$,

$$\begin{aligned} \sum_{i=1}^{N(\varepsilon)} p_i^q &\sim \sum_{i=1}^{K} \left(\frac{\varepsilon}{i}\right)^{q/2} \sim \int_1^{1/(2\varepsilon)} \left(\frac{\varepsilon}{i}\right)^{q/2} di \\ &= \frac{\varepsilon^{q/2}}{q/2 - 1}\,[1 - (2\varepsilon)^{q/2-1}] \\ &\sim \begin{cases} \varepsilon^{q/2} & \text{for} \quad q > 2 \\ \varepsilon^{q-1} & \text{for} \quad q < 2 \end{cases} \end{aligned}$$

Hence, Eq. (1.2) yields

$$d_q = \begin{cases} 1 & \text{for} \quad q < 2 \\ \dfrac{q}{2(q-1)} & \text{for} \quad q > 2 \end{cases}$$

Thus, for $q > 2$ the d_q for f_1 are less than 1, while $d_q = 1$ for $\bar{f}$, and we have demonstrated a change of d_q when making a change of variables for the case $q > 2$.

We can extend this result for $q > 2$ to any $q > 1$, as follows. We assume (and later verify) that there exist maps, $x_{n+1} = g_l(x_n)$, on the interval $[0, 1]$ with $g_l(0) = g_l(1) = 0$ and $g_l(1/2) = 1$, such that, for almost any initial condition $x \in [0, 1]$, an invariant density, f_l is generated where

$$f_l(x) = \frac{\alpha}{[x(1-x)]^{1-1/2l}}$$

and α is chosen so that $\int_0^1 f_l(x)\, dx = 1$. (Our previous example corresponds to the case $l = 1$.) Proceeding as before, one can introduce a change of variables $y(x) = \int_0^x f_l(x')\, dx'$, which transforms the map g_l to the tent map. Upon applying Eq. (2) to f_l and proceeding as in the case $l = 1$, we obtain

$$d_q = \begin{cases} 1 & \text{for} \quad q < \dfrac{2l}{2l-1} \\ \dfrac{q}{2l(q-1)} & \text{for} \quad q > \dfrac{2l}{2l-1} \end{cases}$$

Since $d_q < 1$ for $q > 2l/(2l-1)$ and the quantity $2l/(2l-1)$ can be made arbitrarily close to 1 by increasing l, we have demonstrated a change of variables which changes d_q for any $q > 1$. To find the map g_l giving the invariant density f_l, simply apply the change of variables $y = \int_0^x f_l(x')\, dx'$ to the tent map. In this way, one can readily verify that there is a $2l$-order maximum at $x = 1/2$, i.e., $g_l(x) \sim 1 - (x - 1/2)^{2l}$ near $x = 1/2$. (Our estimates in this section have been made in an intuitive manner without careful attention to magnitude of the errors, but all these arguments can be made rigorous.)

An analysis can also be developed to show that d_q for a Cantor set can be altered for $q > 1$. We omit this treatment here since the case treated in this section is more transparent (although also more special).

3. CHANGE OF d_q UNDER VARIABLE TRANSFORMATIONS FOR $q < 1$

3.1. Capacity: The Case $q = 0$

One example of a compact set whose capacity d_0 and Hausdorff dimension d_H differ is given by a sequence $\{x_n: n = 1, 2,...\}$, which converges to zero sufficiently slowly. We will use this idea in changing the capacity of a set by a change of variable.

Fix $0 < \alpha < 1$ and $c > 0$. Let $x_n = cn^{-\alpha}$ for $n = 1, 2,...$. We will show that the capacity of the set $\{x_n\}$ is at least $1/(\alpha + 1)$. Note that $x_n - x_{n+1} = cn^{-\alpha} - c(n+1)^{-\alpha} \geqslant -(d/dn)(cn^{-\alpha}) = \alpha cn^{-\alpha}$. Suppose we cover the set $\{x_n\}$ with open intervals of length $\varepsilon = c\alpha N^{-\alpha-1}$ for some N. Since $x_n - x_{n+1} \geqslant c\alpha N^{-\alpha-1}$ for $n \leqslant N$, each interval of length $c\alpha N^{-\alpha-1}$ can contain at most one of the points $x_1, x_2,..., x_N$; and $N(\varepsilon) \geqslant N$. Thus

$$\frac{\log N(\varepsilon)}{-\log \varepsilon} \geqslant \frac{\log N}{-\log(c\alpha N^{-\alpha-1})}$$

Taking the limit as $\varepsilon \to 0$ $(N \to \infty)$, we obtain $d_0(\{x_n\}) \geqslant 1/(\alpha + 1)$.

Choose β, $0 < \beta < 1$. We construct a Cantor set in the usual manner. Let $r = 2^{-1/\beta}$ and $\delta = 1 - 2r$. We remove the middle interval of length δ from $[0, 1]$; then the middle intervals of length $r\delta$ from the remaining intervals $[0, r]$ and $[1 - r, 1]$; and so on. For the set C thus constructed both $d_H(C)$ and $d_0(C)$ equal β. [Such sets can result from chaotic attractors (e.g., the baker's-type transformations discussed in Ref. 2).]

The set C contains the sequence $\{r^n: n = 1, 2,...\}$. If we make a change of variable $y = F(x) = (-\log x)^{-\alpha}$, we have $F(r^n) = (-\log r)^{-\alpha} n^{\alpha}$. Since the new set in y coordinates $F(C)$ contains this sequence, we have

$$d_0(F(C)) \geqslant d_0\{[(-\log r)^{-\alpha} n^{-\alpha}]\} \geqslant \frac{1}{\alpha + 1}$$

regardless of the capacity β of the original set C. Thus, if α is chosen sufficiently small, we can guarantee that

$$d_0(F(C)) > d_0(C)$$

3.2. Arbitrary $q < 1$

Let ν be a probability measure on $[0, 1]$ for which $0 < \nu(0, z]$ for $z > 0$, and assume each point has measure 0. Let $0 < q < 1$ and $0 < \alpha < 1$ be fixed numbers. Let $p = 1/(1 - \alpha)(1 - q) > 1$. Let $G_+(x) = 2x + x^p$ and $G_-(x) =$

$2x - x^p$. Since $G_-(x) < G_+(x)$ for $x > 0$, there exists a change of variable $y = F(x)$, which is differentiable for $0 < x \leqslant 1$ with $F(0) = 0$, $F(1) = 1/2$, such that the new measure μ defined by $\mu(0, y] = \mu(0, F(x)] = \nu(0, x]$ satisfies

$$G_-(y) \leqslant \mu(0, y] \leqslant G_+(y), \qquad 0 \leqslant y \leqslant 1/2$$

Furthermore, we can require that F be differentiable except at 0. We will show that $d_q(\mu)$ is at least α, regardless of the dimension of ν.

Choose $\varepsilon > 0$. Let $x_0 = (\varepsilon/2)^{1/p}$. We will assume that ε is sufficiently small that $x_0 \leqslant 1/2$. For $x \leqslant x_0$, we have

$$\begin{aligned}
\mu(x-\varepsilon, x] &= \mu(0, x] - \mu(0, x-\varepsilon] \\
&> G_-(x) - G_+(x-\varepsilon) \\
&\geqslant 2x - x^p - 2x + 2\varepsilon - (x-\varepsilon)^p \\
&\geqslant 2\varepsilon - 2x^p \\
&\geqslant 2\varepsilon - 2x_0^p \geqslant \varepsilon
\end{aligned}$$

Let $p_i = \mu((i-1)\varepsilon, i\varepsilon]$, $i = 1, 2, \ldots, 1/2$. Then

$$\frac{[1/(q-1)] \log \sum p_i^q}{\log \varepsilon} \geqslant \frac{[1/(q-1)] \log \sum' p_i^q}{\log \varepsilon}$$

where $\sum'$ is the sum restricted to $i \leqslant x_0/\varepsilon$. If $i \leqslant x_0/\varepsilon$, then $i\varepsilon \leqslant x_0$, and from the above we have $p_i \geqslant \varepsilon$. Thus

$$\begin{aligned}
\frac{[1/(q-1)] \log \sum p_i^q}{\log \varepsilon} &\geqslant \frac{[1/(q-1)] \log[(1/\varepsilon) x_0 \varepsilon^q]}{\log \varepsilon} \\
&\geqslant \frac{[1/(q-1)] \log(\varepsilon^{1/p - 1 + q} 2^{-1/p})}{\log \varepsilon} \\
&\geqslant \frac{[1/(q-1)] \log(\varepsilon^{\alpha(q-1)} 2^{-1/p})}{\log \varepsilon}
\end{aligned}$$

Since

$$\lim_{\varepsilon \to 0} \frac{[1/(q-1)] \log(\varepsilon^{\alpha(q-1)} 2^{-1/p})}{\log \varepsilon} = \alpha$$

we have $d_q(\mu) \geqslant \alpha$.

4. INVARIANCE OF INFORMATION AND HAUSDORFF DIMENSIONS

4.1. Information Dimension

We now argue that the information dimension of a probability measure ν on $[0, 1]$ is invariant under coordinate changes $y = F(x)$ of the type described in this paper. First we consider the special cases where F' and $1/F'$ are bounded by some integer k. The new measure μ is defined by $\mu(J) = \nu(F^{-1}(J))$ when J is an interval. For any measure ν and $\varepsilon > 0$, let $p_n = \nu([n\varepsilon, n\varepsilon + \varepsilon))$ and write

$$H(\nu, \varepsilon) = -\sum_n p_n \ln p_n$$

Then it can be shown that the bound on the derivative implies[7]

$$|H(\nu, \varepsilon) - H(\mu, \varepsilon)| \leqslant \ln(2k)$$

It follows then that ν and μ have the same information dimension.

We now consider the case in which the derivative is unbounded or is 0 on a finite set. Choose S_δ to be an open set with $\nu(S_\delta) = \delta$ where S_δ consists of a finite collection of intervals, and assume that on $[0, 1] - S_\delta$, the change of variables F has $|F'|$ and $1/|F'|$ bounded.

Let ν_0 and ν_1 be any two probability measures having the closure of their supports disjoint except possibly at a finite number of points, and define ν_α for $\alpha \in [0, 1]$ by

$$\nu_\alpha(E) = (1 - \alpha)\, \nu_0(E) + \alpha \nu_1(E)$$

for all measurable sets E. As can be seen from the definition of d_I it follows[3] that

$$d_I(\nu_\alpha) = (1 - \alpha)\, d_I(\nu_0) + \alpha d_I(\nu_1) \tag{4.1}$$

For any measurable set S and probability measure ν we define a new probability measure ν_S by

$$\nu_S(E) = \nu_S(S \cap E)/\nu_S(S)$$

In particular, $\nu_S(S) = 1$ and $\nu_S([0, 1]) = 0$. Now applying these ideas to μ, ν, and S_δ we note that F' and $1/F'$ are bounded on $S_\delta^c (= [0, 1] - S_\delta)$. Thus

$$d_I(\nu_{S_\delta^c}) = d_I(\mu_{F(S_\delta^c)}) \tag{4.2}$$

Notice that for any measurable set E

$$\nu(E) = (1-\delta)\,\nu_{S_\delta^c}(E) + \delta \nu_{S_\delta}(E)$$

so ν_δ, the measure on the right-hand side, equals ν, and (4.1) gives

$$d_I \nu = (1-\delta)\, d_I \nu_{S_\delta^c} + \delta d \nu_{S_\delta} \tag{4.3}$$

A corresponding μ formula holds using $F(S_\delta)$. Hence, taking the difference and using (4.3) for both μ and ν yields

$$\begin{aligned} |d_I \nu - d_I \mu| &\leqslant (1-\delta)\,|d_I \nu_{S_\delta^c} - d_I \mu_{F(S_\delta^c)}| \\ &\quad + \delta\,|d_I \nu_{S_\delta} - d_I \mu_{F(S_\delta)}| \leqslant \delta \end{aligned}$$

The last inequality follows from (4.2) and the fact that $d_I \nu_{S_\delta}$ and $d_I \mu_{F(S_\delta)}$ must be between 0 and 1 since we are considering measures on $[0, 1]$. Since δ is arbitrary, it follows that $d_I \nu = d_I \mu$, which is what we wished to prove.

While the result (4.1) seems like a bizarre property for a dimension to satisfy, we would expect it to have little relevance to attractor theory since when ν is a natural measure associated with an ergodic attractor and S is a finite union of intervals, we expect $d_I(\nu_S)$ to be independent of the choice of S.

4.2. Hausdorff Dimension

We review the definition of the Hausdorff dimension and the Hausdorff measure of a set $V \subset R^n$. Let $\alpha \geqslant 0$ and $\varepsilon > 0$. We let $\{C_j : j = 1, 2,...\}$ be a collection of disks such that $V \subset \bigcup C_j$. Unlike the definition of capacity, we allow the disks C_j to vary in size, with diameters ranging from 0 to ε. We define

$$m_\alpha(\varepsilon, V) = \inf \sum_j (\operatorname{diam} C_j)^\alpha$$

choosing the set of disks $\{C_j\}$ so that the sum on the right is as small as possible. As ε decreases, we have fewer collections of disks $\{C_j\}$ to choose from, and thus $m_\alpha(\varepsilon, V)$ cannot decrease. We let

$$m_\alpha V = \lim_{\varepsilon \to 0} m_\alpha(\varepsilon, V)$$

Then $m_\alpha V$ is the two-dimensional Hausdorff measure of V. When α equals an integer $m \leqslant n$, $m_\alpha V$ is essentially the m-dimensional volume of V. When $\alpha = 0$, $m_\alpha V$ is the number of points in V.

For any set V, there is a number $d_H(V)$ such that $m_\alpha V = 0$ for $\alpha > d_H(V)$ and $m_\alpha V = \infty$ for $\alpha < d_H(V)$. We call $d_H(V)$ the Hausdorff dimension of V. For $\alpha = d_H(V)$, $m_\alpha V$ may take any value from 0 to ∞, inclusive, depending on V.

We let F be a mapping from R^n to R^n (i.e., F is a change of coordinates) and ask under what circumstances $d_H(F(V))$ is greater than $d_H(V)$. If F is differentiable at x then there is an integer n such that

$$\|F(x) - F(y)\| < n\,\|x - y\|$$

For all y with $\|x - y\| < 1/n$. Thus we may write

$$V = W \cup V_1 \cup V_2 \cup V_3 \cup \cdots$$

where $W = \{x \in V\colon\ F$ is not differentiable at $x\}$ and $V_n = \{x \in V\colon \|F(x) - F(y)\| < n\,\|x - y\|$ for all y with $\|x - y\| < 1/n\}$. Suppose $d_H(V) < d_H(F(V))$ and let α be such that $d_H(V) < \alpha < d_H(F(V))$. Then we have $m_\alpha V = 0$, while $m_\alpha F(V) = \infty$.

Suppose $\{C_{nj}\colon j = 1, 2,...\}$ is a collection of disks whose union contains V_n, with diameters between 0 and ε, where $0 < \varepsilon < 1/2n$.

Let $x_j \in V_n \cap C_{nj}$. Then $F(C_{nj})$ is contained in the disk D_{nj} centered at $F(x_j)$ and with diameter $2n$ diam C_{nj}, and $F(V_n)$ is contained in the union of the disks D_{nj}. Moreover,

$$\sum_j (\text{diam } D_{nj})^\alpha = 2^\alpha n^\alpha \sum_j (\text{diam } C_{nj})^\alpha$$

Since $m_\alpha V = 0$ and $V_n \subset V$, we have $m_\alpha(\varepsilon, V_n) = 0$; in other words, the right side of the equality above can be made as small as we please by the right choice of the disks C_n. But then the left side can be made as small as we please by the right choice of the disks D_n. Thus

$$m_\alpha(2n\varepsilon, F(V_n)) = 0$$

Thus

$$m_\alpha F(V_n) = \lim_{\varepsilon \to 0} m_\alpha(2n\varepsilon, F(V_n)) = 0$$

We will suppose that $m_\alpha F(W)$ is finite. This is true whenever the set W itself is finite, as well as when $d_H(F(W)) < \alpha$. Then we have

$$m_\alpha F(V) \leqslant m_\alpha F(W) + m_\alpha F(V_1) + m_\alpha F(V_2) + \cdots < \infty$$

But by assumption, $m_\alpha F(V) = \infty$. Thus we must have $d_H F(V) \leqslant d_H V$. We may apply the same argument to F^{-1} to obtain

$$d_H F(V) = d_H V$$

Hence, the Hausdorff dimension is invariant under a wide class of changes of variables, including all changes of variables which are differentiable except at a finite set of points.

ACKNOWLEDGMENTS

This work was supported under the Department of Energy (Basic Energy Sciences) Grant No. Doe-AT05-83ER12026, under Air Force Office of Scientific Research Grant No. 81-0217, and under National Science Foundation Grant No. MCS 81-17967.

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